Suggested Solutions to Midterm 1

MATH 2040 A 2015-16 2nd Semester.

Q1. **(True or False)** Please circle the correct answer. Each question worths 0.5 points. (You do not have to explain your answer.)

(i) The space $C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \text{ continuous}\}$, with usual addition and scalar multiplication, is a finite dimensional vector space over $\mathbb{R}$.

$\because$ not finite dimensional. TRUE **FALSE**

(ii) If W_1, W_2 are subspaces of a vector space V, then $W_1 \cap W_2$ is a subspace of V.

TRUE FALSE

(iii) If S is a linearly independent subset of a vector space V, and $v \in V$ is a vector such that $S \cup \{v\}$ is linearly dependent, then $v \in span(S)$.

TRUE FALSE

(iv) An $n \times n$ matrix A has at most n distinct eigenvalues.

TRUE FALSE

(v) Let $A \in M_{n\times n}(\mathbb{R})$ and $v, w \in \mathbb{R}^n$ such that $Av = 2v$ and $Aw = 5w$. Then $\{v, w\}$ is linearly independent.

$\because$ V or W could be zero vector TRUE **FALSE**

(vi) If $A, B \in M_{n\times n}(\mathbb{R})$ such that $\det(A) = 1$ and $\det(B) = 3$, then $\det(A + B) = 4$.

$\because \det(A+B) \neq \det A + \det B$ TRUE **FALSE**

(vii) If an invertible matrix $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable, then A^{-1} is also diagonalizable.

TRUE FALSE

(viii) Let $T : V \to V$ be a linear operator on a finite dimensional vector space V, and let $v_1, v_2 \in V$. Suppose W_1 and W_2 are the T-cyclic subspaces generated by v_1 and v_2 respectively. If $W_1 = W_2$, then $v_1 = v_2$.

E.g. $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotation by $90°$. TRUE **FALSE**

take $v_1 = e_1$, $v_2 = e_2$

$W_1 = W_2 = \mathbb{R}^2$ but $e_1 \neq e_2$.

Q2. **(Short Questions)** Each question worth 1 point. (You do not have to explain your answer.)

(i) If the characteristic polynomial of $A \in M_{3\times 3}(\mathbb{R})$ is $f(\lambda) = -(\lambda-2)^2(\lambda+3)$, what are the eigenvalues of the transpose A^t?

Answer: 2, -3

(ii) Let $A \in M_{20\times 25}(\mathbb{R})$. If $\dim\{Ax : x \in \mathbb{R}^{25}\} = 5$, what is the nullity of A?

Answer: 20

(iii) Write down two diagonalizable (over $\mathbb{R}$) matrices $A, B \in M_{2\times 2}(\mathbb{R})$ such that $A+B$ is not diagonalizable (over $\mathbb{R}$).

Answer: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

(iv) Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$. Write down an invertible matrix $Q \in M_{3\times 3}(\mathbb{R})$ which is not the identity matrix such that $Q^{-1}AQ$ is diagonal.

Answer: $Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(v) Suppose $\{u, v, w\} \subset \mathbb{R}^3$ is a basis for $\mathbb{R}^3$. Find a constant $a \in \mathbb{R}$ such that $\{u - v, u + v + w, -2u + v + aw\}$ is not a basis for $\mathbb{R}^3$.

Answer: $a = -\frac{1}{2}$

(vi) Let $A = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. Express A^{-1} as a linear combination of the matrices I and A. (*Hint: Cayley-Hamilton Theorem*)

Answer: $A^{-1} = \frac{1}{2}(A - 7I)$

Q.3 (a) (2 points) Let $P \subset \mathbb{R}^3$ be the plane given by the equation $2x - y + z = 0$. Find a basis β for the subspace P.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in P \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -2x+y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for P.

(b) (4 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by reflection about the plane P in (a). Show that T is diagonalizable and find an eigenbasis for T.

$$\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Let $\gamma = \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}$ a basis for $\mathbb{R}^3$.

Then $[T]_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Hence T is diagonalizable and γ is an eigenbasis for T.

(c) (4 points) Find $[T]_{\beta'}$ where β' is the standard basis for $\mathbb{R}^3$.

Let γ as in (b). Then

$$[T]_{\beta'} = [I]_{\gamma}^{\beta'} [T]_{\gamma} [I]_{\beta'}^{\gamma}$$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -2 & 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 5/6 & 1/6 \\ 1/3 & -1/6 & 1/6 \end{pmatrix}$$

$$= \begin{pmatrix} -1/3 & 2/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{pmatrix}$$

Q.4 (a) (3 points) Let W_1, W_2 be T-invariant subspaces of a linear operator $T: V \to V$ on a vector space V. Prove that the sum $W_1 + W_2$ is also a T-invariant subspace.

First of all, $W_1 + W_2$ is a subspace of V. To show it is T-invariant, let $w \in W_1 + W_2$. Then $w = w_1 + w_2$ for some $w_1 \in W_1$, $w_2 \in W_2$.

Since W_1, W_2 are T-invariant, we have $Tw_1 \in W_1$ and $Tw_2 \in W_2$.

It follows that $Tw = Tw_1 + Tw_2 \in W_1 + W_2$.

Hence $W_1 + W_2$ is T-invariant.

(b) (3 points) Let $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. Is A diagonalizable over $\mathbb{R}$? Justify your answer.

The characteristic polynomial of A is

$$\det\begin{pmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$$

A has two distinct eigenvalues, namely 1 and 2, so A is diagonalizable.

Q.5 (a) (2 points) Let $A, B \in M_{n\times n}(\mathbb{C})$. Prove that if B is invertible, then there exists a scalar $c \in \mathbb{C}$ such that $A + cB$ is not invertible.

If B is invertible, then $AB^{-1} \in M_{nn}(\mathbb{C})$ and

$\exists\, c \in \mathbb{C}$ s.t. $AB^{-1} + cI$ is not invertible. since eigenvalues always exist over $\mathbb{C}$.

Then $\det(A+cB) = \det(AB^{-1}+cI)\cdot\det(B) = 0$.

Hence $A+cB$ is not invertible.

(b) (1 points) Give an example of nonzero matrices $A, B \in M_{2\times 2}(\mathbb{C})$ such that $A + cB$ is invertible for *all* $c \in \mathbb{C}$. No justification is needed.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(c) (1 points) Given an example of two matrices $A, B \in M_{2\times 2}(\mathbb{R})$ such that B is invertible but $A + cB$ is invertible for all $c \in \mathbb{R}$. No justification is needed.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

—END OF MIDTERM 1—